# IV Semester M.Sc. Degree Examination, June/July 2014
## (RNS) (2012-13 and Onwards)
## MATHEMATICS
## M-401 : Measure and Integration

Time: 3 Hours Max. Marks: 80

*Instructions*: i) Answer **any five** questions choosing atleast **two** from **each** Part.

ii) **All** questions carry **equal** marks.

PART – A

1. a) Define $\sigma$ – algebra. Prove that outer measure is finitely additive over disjoint sets. 4

b) State and prove countable additive property for Lebesgue measurable sets. 4

c) Prove that the interval $(a, \infty)$ is measurable and deduce that every Borel set is measurable. 8

2. a) Construct Cantor ternary set from $[0, 1]$. Show that Cantor's ternary set is uncountable but has measure zero. 8

b) Show that $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^* A_n$ for any countable collection $\{A_n\}$ of sets in R. Deduce that measure of any countable set is zero. 8

3. a) Let f be measurable function and B be a Borel set. Then prove that $f^{-1}(B)$ is a measurable set. 7

b) Let $\{f_n\}$ be a sequence of measurable functions defined on a set of finite measure and let f be a real valued measurable function on E such that $\{f_n(x)\}$ converges to f(x) for all $x \in E$. Then prove that given $\varepsilon > 0$ and $\delta > 0$ there is a measurable set $A \subset E$ with $m(A) < \delta$ and a positive integer N such that for $x \in E - A$, we have $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$. 6

c) If f is a measurable function on E, then prove that $|f|$ is also measurable. 3

4. a) State and prove Egroff's theorem. 6
   b) Show that a continuous function over a measurable set E is a measurable function. 3
   c) Let E be a Lebesgue measurable set with finite measure. For a given $\varepsilon > 0$, prove that there exists a finite union 'U' of open intervals such that $m(E \Delta U) < \varepsilon$ where $E \Delta U = (E - U) \cup (U - E)$. 7

PART – B

5. a) If f and g are bounded measurable function defined on a set is finite Then
   i) $\int_E af + bg = a\int_E f + b\int_E g$
   ii) If f = g almost everywhere, then $\int_E f = \int_E g$
   iii) If $f \leq g$ almost everywhere $\int_E f \leq \int_E g$ and $\left|\int_E f\right| \leq \int_E |f|$. 6
   b) State and prove Monotone convergence theorem. 6
   c) Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable function and let $f = \sum_{n=1}^{\infty} U_n$ a.e. on E then, prove that $\int_E f = \sum_{n=1}^{\infty} \int_E U_n$. 4
6. a) Let f and g be two non-negative measurable functions and if f is integrable over E and $g(x) \leq f(x)$ on E then prove that g is also integrable over E and $\int_E f - g = \int_E f - \int_E g$. 4
   b) If f is an integrable function on a measurable set E, then prove that $|f|$ is also integrable and $\int_E |f| dx \geq \left|\int_E f\right|$ 5
   c) Define a function of bounded variation. If f is a bounded variation on [a, b], then prove that
   $T_a^b = P_a^b + N_a^b$ and
   $f(b) - f(a) = P_a^b - N_a^b$ 7
7. a) Establish Vitali covering lemma. 9
   b) Define a summable series. Prove that a normed linear space X is complete if and only if every absolutely summable series in X is summable. 7
8. a) Assuming Holder's inequality in $L^p$ $(1 \leq p \leq \infty)$, prove Minkowski's inequality. 8
   b) State and prove Riesz-Fischer theorem. 8